# Birla Institute of Technology & Science, Pilani

**MATH F425 (Numerical Linear Algebra)**
**Second Semester 2022-2023**
**Comprehensive Examination (Closed Book)**

**Time: 90 Minutes** **Date: May 06, 2023 (Saturday)** **Max. Marks: 23**

1. Notations and symbols have their usual meaning.
2. Start new question on fresh page. **Moreover, answer each subpart of a question in continuation.**
3. Write **END** at the end of the last attempted question.

1. Fit a straight line and a quadratic to the data points $(1,1),(2,5),(1,-2),(3,4)$ by using least square and compare the results when $x=3$. **[3]**

2. If possible, find the optimal choice for the relaxation factor $\omega$ by using the following matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

with the help of Jacobi's iteration matrix. Hence, find the spectral radius of the SOR iteration matrix. **[3]**

3. Consider the data matrix

$$X = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

Determine the new data set corresponding to the first principal component analysis. **[3]**

4. Consider the linear system of equation $4x - y = 1, \quad -x + 2y = 0$. Do two steps of steepest descent and conjugate gradient methods to solve the system using initial approximation $\mathbf{X}^{(0)} = [0,0]^T$. What do you observe special about CG method. **[7]**

5. Consider a 4-dimensional tensor $\mathcal{X} \in \mathbb{R}^{3\times4\times4\times5}$. Write the tensor train (TT) format of $\mathcal{X}$ and using a sequence of singular value decomposition as well as reshape function, find the core tensors of TT with reduced rank 2 for each core tensor. Discuss the storage complexity for full tensor and the tensor in TT format. **[4]**

6. If $\mathcal{X} \in \mathbb{R}^{2\times3\times2}$ such that the two slices are

$$X_1 = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}.$$

Write the formula to compute the tensor with matrix multiplication with mode-3 where $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$ and hence, write the slice of the resulting tensor with fixing the last index as 1. **[3]**

*****END*****

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1. Determine all values of $a, b$ and $c$ for which the matrix $A = \begin{pmatrix} 4 & a+b-2c & 0 \\ -1 & 3 & b-c \\ -a+b+c & 1 & b-2c \end{pmatrix}$ can be reduced into the form of $LL^T$ by using Cholesky method. Find $L$. **[3]**

2. Is LU decomposition of a matrix $A$ unique always? If yes, prove it, else provide a counter example by taking a matrix $A$ of size $2 \times 2$ or of your choice. **[2]**

3. Consider a linear system $Ax = b$, where $A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ and $b = [1, 1, 2]^T$. Suppose that $a_{11} = 2$ in $A$ is replaced by $a_{11} = 1.99$ and $b$ is changed to $b_1 = [1, 1, 1.99]^T$. How large a relative change can this change produce in the solution to $AX = b$. **[3]**

4. (a) If $\lambda_i$ for $i = 1, \ldots, n$ be the eigenvalues of matrix $A$ and $\|A\| < 1$, then prove or disprove that

$$\|(I - A)^{-1} - I\| \leq \frac{\|A\|}{1 - \|A\|}.$$

   (b) Let $H = I - \frac{2uu^T}{u^T u}$ be a Householder matrix with $u \in \mathbb{R}^n$. Then show that
   (i) $Hu = -u$ (ii) $Hv = v$ if $v^T u = 0$.

   (c) Show that the floating point computation of the inner product of two vectors is backward stable.

   (d) Determine whether the following statements are True or False:
   (i) If the determinant of a matrix $A$ is small, then $A$ must be close to a singular matrix.
   (ii) If the zeros of a polynomial are all distinct, then they must be well-conditioned.
   (iii) The factorization $PAQ = LU$ in GECP needs $\frac{2n^3}{3}$ flop-count $+(\mathcal{O}(n^2)$ comparisons).
   (iv) $\|AB\|_2 \leq \|A\|_2 \frac{1}{\|B^{-1}\|_2}$.
   (v) For positive definite matrices, the orthogonal decomposition and the SVD coincide. **[2+2+2+2]**

5. Find the least square solution of $AX = B$ where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 3 & 0 \\ 3 & 0 \\ 5 & 2 \end{pmatrix}$$

by converting into normal equation form and then solving resulting system of equations using LU decomposition with partial pivoting method. Under what condition, such a solution is unique. **[6]**

*****END*****