BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI,
K K BIRLA GOA CAMPUS
Open Book, Open Laptop, Internet NOT allowed

Subject Name: MATH F432 - Applied Statistical Methods Date: 30 Dec 2022
Examiner Name: Sravan Danda Marks: 40
Duration: 2.5 hours (2 PM - 4:30 PM)

Attempt all questions. Marks corresponding to each question is highlighted in bold within square braces at the end of the question. You are allowed to carry any material on your laptops. However, you would not be allowed to use the internet. In case of ambiguities in any of the questions, clearly state your assumptions and attempt the question(s).

1. It is desired to estimate the value of π. Consider the regions in $\mathbb{R}^2$ defined by $S_1 = \{(x, y) : -1/\sqrt{2} \leq x, y \leq 1/\sqrt{2}\}$, $C = \{(x, y) : x^2 + y^2 \leq 1\}$, and $S_2 = \{(x, y) : -1 \leq x, y \leq 1\}$. Suppose $X \sim Uniform(S_2)$ and $Y_1, \cdots, Y_n, \cdots \sim Uniform(C)$ are all independent of each other.

   (a) T_1 is given by: conditional on a point picked uniformly at random from S_2, indicator function of the point lying within C i.e.
   $$T_1 = I_{\{X \in C\}}|X \tag{1}$$
   Find a_1 and $b_1$such that $a_1T_1+b_1$ is an unbiased estimator of π where $a_1, b_1 \in \mathbb{R}$ are constants.

   (b) T_2 is given by: conditional on the points sampled uniformly at random and independently from C, the minimum number of points required to sample until a point lies within S_1
   $$T_2 = min\{n : Y_n \in S_1\}|Y_1, \cdots, Y_n \tag{2}$$
   Find a_2 and b_2 such that $a_2T_2+b_2$ is an unbiased estimator of π where $a_2, b_2 \in \mathbb{R}$ are constants.

   (c) Check which of the two unbiased estimators $a_1T_1 + b_1$ and $a_2T_2 + b_2$ has a smaller variance.

   [(1+1)+(1+1)+3]

2. Suppose we are interested in estimating
   $$\theta = E[(1 - U^2)] \tag{3}$$
   using simulation where $U \sim Uniform(0, 1)$.

   (a) Find the variance of the naive estimator $T_1 = 1 - U^2$

   (b) Using U as a control variate, obtain the smallest variance among all the estimators from the class $\mathcal{T}_2 = \{T_1 + c\left(U - \frac{1}{2}\right) |c \in \mathbb{R}\}$.

   (c) Using U^3 as a control variate, obtain the smallest variance among all the estimators from the class $\mathcal{T}_3 = \{T_1 + c\left(U^3 - \frac{1}{4}\right) |c \in \mathbb{R}\}$.

   (d) Using (b) and (c) or otherwise identify which among U and U^3 is a better option as a control variate for $1 - U^2$ to estimate θ.

   [2+2+2+1]

3. Suppose $X_1, \cdots, X_n$ are independent and identically distributed $Bernoulli(p)$ random variables where $p = 0.1$. Let S_1 denote the sum of the n independent random variables i.e. $S_1 = \sum_{i=1}^{n} X_i$. Let $n = 10$. We are interested in estimating the probability that $P\{S_1 \geq 9\}$ using simulation. The first

estimator is given by simulating $X_1, \cdots, X_n$ from $Bernoulli(p)$ and using the indicator function i.e.

$$T_1 = I_{\{S_1 \geq 9\}} \tag{4}$$

The second estimator is based on importance sampling and is constructed using a tilted density. $Y_1, \cdots, Y_n$ are simulated from $Bernoulli(q)$ with $q = 0.9$ and denote $S_2 = \sum_{i=1}^{n} Y_i$. The estimator is given by

$$T_2 = I_{\{S_2 \geq 9\}} \left(\frac{1}{81}\right)^{S_2} 9^{10} \tag{5}$$

(a) Compute the variance of T_1 (you do not have to simplify the numbers).

(b) Compute the maximum value of T_2.

(c) Show that if a random variable Z satisfies $P\{0 \leq Z \leq a\} = 1$ for some constant $a > 0$ the following hold

i. $E[Z^2] \leq aE[Z]$.

ii. $Var(Z) \leq E[Z](a - E[Z])$

iii. $Var(Z) \leq \frac{a^2}{4}$

(d) Using (b) and (c) or otherwise, argue that $\frac{Var(T_2)}{Var(T_1)} < 10^{-3}$

[1+1+(1+1+1)+1]

4. Suppose the joint density of X, Y, Z is given by

$$f(x, y, z) = Ce^{-(x+y+z+axy+bxz+cyz+dxyz)} I_{\{x>0, y>0, z>0\}} \tag{6}$$

where a, b, c, d are specified non-negative constants, and C does not depend on x, y, z. Using Gibbs sampling or otherwise simulate a random vector (X, Y, Z) from the given joint density.

[10]

5. Consider the function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ given by

$$f(x, y) = \frac{1}{xy^2} + xy^2 \tag{7}$$

(a) Argue that f attains its global minimum i.e. there exists $(\hat{x}, \hat{y}) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $f(\hat{x}, \hat{y}) \leq f(x, y)\ \forall (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$. Find the minimum value attained by f. Describe the set of minimizers i.e. $\mathcal{M} = \{(\hat{x}, \hat{y}) \in \mathbb{R}^+ \times \mathbb{R}^+ : f(\hat{x}, \hat{y}) \leq f(x, y)\ \forall (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+\}$.

(b) Argue that for $x_k, y_k \in \mathbb{R}^+$, $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ given by

$$g(x, y | x_k, y_k) = \frac{y_k^2}{2x_k} x^2 + \frac{x_k^2}{3y_k^2} \frac{1}{x^3} + \frac{x_k}{2y_k^2} y^4 + \frac{2y_k}{3x_k} \frac{1}{y^3} \tag{8}$$

is a majorizer of f at (x_k, y_k) i.e.

$$g(x_k, y_k | x_k, y_k) = f(x_k, y_k) \tag{9}$$

and

$$g(x, y | x_k, y_k) \geq f(x, y)\ \forall (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \tag{10}$$

(c) Suppose a majorization-minimization algorithm is applied using g as the majorizer. For every minimizer $(\hat{x}, \hat{y}) \in \mathcal{M}$, argue that if the initial point is chosen as $(x_0, y_0) = (\hat{x}, \hat{y})$ then the algorithm converges to $(\hat{x}, \hat{y})$ within a single iteration i.e. $(x_1, y_1) = (x_0, y_0)$.

(d) Suppose a majorization-minimization algorithm is applied using g as the majorizer. Argue that irrespective of the choice of the initial point (x_0, y_0) the algorithm always converges to a global minimum.

[(1+1+1)+(1+2)+2+2]