BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE, PILANI

Department of Mathematics, First Semester, 2022-23

Comprehensive exam (Open Book)

Numerical Solutions to Ordinary Differential Equations (MATH F444)

27/12/2022

Max. Time : 180 mins **Max. Marks: 40**

Name :... **ID No. :**..

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1. Consider the following Runge-Kutta method with Butcher tableau

$$\begin{array}{c|cc} 1/3 & 5/12 & -1/12 \\ \alpha & \beta & 1/4 \\ \hline & \gamma & 1/4 \end{array}$$

(i) Find the values α, β and γ such that this numerical scheme has highest possible accuracy. [4]

(ii) Discuss the stability of numerical scheme for these obtained values of α, β and γ. [6]

2. Given that the characteristic polynomial $\rho(\xi) = \xi^2 - 1$, find the corresponding explicit LMM. Also discuss the $0-$stability of this numerical scheme. [3+3]

3. Recall the system of ODE for Timoshenko's beam from the mid-semester exam, which had the form of

$$\frac{d^2u(x)}{dx^2} - \frac{d\gamma(x)}{dx} = 0, \; \frac{d^2\gamma(x)}{dx^2} - \alpha\frac{du(x)}{dx} = 0 \text{ for } x \in [0,1].$$

where α is a constant.

(i) Check whether or not

$$Y(x) = \begin{pmatrix} 1 & \frac{\sinh(\sqrt{\alpha}x)}{\sqrt{\alpha}} & 0 & \frac{\cosh(\sqrt{\alpha}x)-1}{\alpha} \\ 0 & \cosh(\sqrt{\alpha}x) & 0 & \frac{\sinh(\sqrt{\alpha}x)}{\sqrt{\alpha}} \\ 0 & \cosh(\sqrt{\alpha}x)-1 & 1 & \frac{\sinh(\sqrt{\alpha}x)}{\sqrt{\alpha}} \\ 0 & \sqrt{\alpha}\sinh(\sqrt{\alpha}x) & 0 & \cosh(\sqrt{\alpha}x) \end{pmatrix}$$

is the fundamental solution matrix for the given system of ODEs (don't try to derive it, rather use the definition of the fundamental solution matrix). [5]

(ii) For $\alpha = 1$, discuss the uniqueness and existence for BVP with the following boundary conditions:

(a) $u(0) = 1$, $u'(0) = 0$, $\gamma(1) = 0$, and $\gamma'(1) = 0$. [3]

(b) $u'(0) = 1$, $\gamma(0) + u'(0) = 0$, $\gamma(1) = 0$, and $\gamma'(1) = 0$. [2]

4. Find the index of the following system of DAEs (motion of a pendulum of a fixed length l)

$$\begin{aligned} x'(t) &= u(t), \\ y'(t) &= v(t), \\ u'(t) &= -\frac{2\lambda(t)}{m}\frac{x(t)}{\sqrt{x(t)^2+y(t)^2}}, \\ v'(t) &= g - \frac{2\lambda(t)}{m}\frac{y(t)}{\sqrt{x(t)^2+y(t)^2}}, \\ \sqrt{x(t)^2+y(t)^2} &= l. \end{aligned}$$

The unknowns are the coordinates $x(t)$, $y(t)$, their velocities $u(t)$ and $v(t)$, and the force exerted by the rod is $\lambda(t)$, while m (mass) and g (gravity) are known constants. [4]

5. Consider the two-point boundary value problem

$$((1+x)u')' = 0, \quad 0 < x < 1; \quad u(0) = 0, \quad u'(1) = 1.$$

(i) Write the weak form of the given BVP. [2]

(ii) Divide the interval $0 < x < 1$ into 3 sub-intervals of equal length $h = 1/3$ and construct the stiffness matrix M and the load vector b. [6]

(iii) Find u_0, u_1, u_2 and u_3. [2]