BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE, PILANI

Department of Mathematics

First Semester, 2022-23

Numerical Solutions to Ordinary Differential Equations (MATH F444)

04/11/2022

Max. Time : 90 mins **Max. Marks: 35**

Name :... **ID No. :...**

1. Consider the IVP $y'(t) = \sqrt{y(t)}$; $y(0) = 0$, which has a solution $y = t^2/4$. Apply the Euler-forward scheme, i.e.,

$$y_{n+1} = y_n + hf(t_n, y_n)$$

to the given IVP.

(i) Compute y_1 and y_2 with $h = 0.1$.

(ii) Does this numerical scheme converges to the solution $y = t^2/4$, as $h \to 0$? If not why not (explain)? [2+2]

2. Consider the following implicit Runge-Kutta method with Butcher tableau

$$\begin{array}{c|cc} \alpha & \alpha & 0 \\ 1-\alpha & 1-2\alpha & \alpha \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

(i) Find the amplification factor $R(z)$ for this numerical scheme, hence discuss the absolute stability (A-stability) of this numerical scheme. [5]

Recall the inverse of a 2×2 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(ii) What is the minimum value of α for which the method is always A-stable. [2]

(b) Find the order of this numerical scheme. [3]

3. (i) Discuss the consistency, accuracy and zero-stability for the following three-step method [4]

$$y_{n+2} - y_{n+1} = h\left(-\frac{1}{12}f_n + \frac{8}{12}\beta_1 f_{n+1} + \frac{5}{12}f_{n+2}\right).$$

(ii) What is the maximal order of an A-stable linear multistep method? [1]

4. Consider the following family of Runge–Kutta methods:

$$y_n = y_{n-1} + h\left(ak_1 + bk_2\right),$$

where

$$\begin{aligned} k_1 &= f\left(x_{n-1}, y_{n-1}\right), \\ k_2 &= f\left(x_{n-1} + \frac{1}{3}h, y_{n-1} + \beta h k_1\right). \end{aligned}$$

(i) Construct the Butcher tableau for the given scheme. Is this an implicit scheme or an explicit scheme? [2]

(ii) Determined the parameters a, b, and β so that the method has highest possible order. [3]

5. Consider a three-point stencil for the finite difference method

$$u'(x_0) = au(x_0) + bu(x_0 + h) + cu(x_0 + 2h),$$

Find a, b and c such that the numerical scheme is third order. [3]

6. The system of (homogeneous) ODE for Timoshenko's beam, which has the form of

$$\frac{d^2u(x)}{dx^2} - \frac{d\gamma(x)}{dx} = 0, \quad \frac{d^2\gamma(x)}{dx^2} - \alpha\frac{du(x)}{dx} = 0 \text{ for } x \in [0, 1].$$

where α is a constant (which depends on the material properties of the beam).

(i) Transform it to the first-order system of ordinary differential equation. [2]

(ii) Find the corresponding fundamental matrix. [6]