Birla Institute of Technology & Science, Pilani

First Semester 2023-2024, MATH F444 (Numerical Solutions of ODEs)

Comprehensive Examination

Time: 180 Min. **Date: December 09, 2023 (Saturday)** **Max. Marks: 45**

1. Question paper consists of Part–A (Closed Book) for 100 minutes and Part–B (Open Book) for 80 minutes. Attempt questions of Part–A and Part–B in two separate answer sheets.
2. Write Part–A and Part–B on top right corner of the two answer sheets provided.
3. Part–B answer sheet will be given only after submission of Part–A answer sheet. Early submission of Part–A is allowed.
4. Use 3 significant digit arithmetic with rounding whenever it is not specified.

Max Marks: 25 Part-A (Closed Book) Time: 100 Minutes

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1. Convert the given BVP $y^{''} + y = 0$ into system of ODEs. For $\cos b = 1$ with periodic boundary conditions $y(0) = y(b)$ and $y^{'}(0) = y^{'}(b)$, find whether BVP will have a unique solution or not. **[4]**

2. (a) Find the condition on h such that the second order finite difference formulation is stable for solving the problem **[3+4]**

$$y^{''} + 10y^{'} = 0.$$

   (b) By using a second order finite difference scheme, express in matrix form ($AY = b$) a discrete approximation to the BVP

$$y'' + 2y' - 3y = 1 + x$$

   with boundary conditions $y(0) - y'(0) = 1$ and $y'(1) = 2$ with $h = 1/3$.

3. Consider the boundary value problem **[6]**

$$-((1 + x^2)u')' = x, \quad 0 < x < 2; \quad u(0) = 0, u'(2) = 2.$$

   Divide the interval $0 < x < 2$ into 3 subintervals of equal length $h = 2/3$ and hence, compute the stiffness matrix A and the load vector b by using finite element method.

4. Use the shooting method to solve the boundary value problem **[5]**

$$y^{''} = 2\,y\,y^{'}, \quad 0 < x < 2$$

   with $y(0) = 0.5, y(2) = 1$. To solve the IVP, use 2nd order Runge-Kutta scheme with $h = 0.50, y^{'}(0) = 1$. Use one iteration of Newton-Raphson's method to find the updated slope to solve the IVP.

5. For the fixed l and m, find the index of the following DAEs **[3]**

$$x'(t) = u(t), \quad y'(t) = v(t), \quad u'(t) = -\frac{2\lambda(t)x(t)}{ml}, \quad v'(t) = 1 - \frac{2\lambda(t)y(t)}{ml}, \quad x^2 + y^2 = l^2.$$

Max Marks: 20 Part-B (Open Book) Time: 80 Minutes

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1. Consider the numerical scheme given as $y_{n+1} = y_n + hf(\frac{t_n+t_{n+1}}{10}, \frac{\theta(y_n+y_{n+1})}{2})$. **[2+2]**

   (a) Considering the test equation $y' = \lambda y$, what values of θ provide the second order accuracy of the scheme.

   (b) If $f(t,x)$ satisfies a Lipschitz condition in x with Lipschitz constant L, then find the Lipschitz constant for $f(t_n + 2h, x_n + \frac{h}{3}f(t_n, x_n))$ in the second argument.

2. An approximate solution to the problem $\frac{dy}{dx} = f(x,y)$ with $y(x_n) = y_n$ at $x = x_n + h$ is given by the Runge-Kutta formula as **[3]**

$$y(x_n + h) = y_n + 1/6(k_1 + 4k_2 + k_3)$$

   where $k_1 = hf(x_n, y_n), k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}), k_3 = hf(x_n + h, y_n + R)$ and $R = hf(x_n + h, y_n + k_1)$. Find the order of local error with proper justification.

3. For the solution of problem $y^{'} = f(x,y)$ with $y(x_0) = y_0$, the following method is used **[3]**

$$y_j = y_{j-1} + \alpha h y_j^{'} + h^2 \left(\frac{1}{2} - \alpha\right) y_j'' + h(1-\alpha)y_{j-1}'.$$

   Find the value of α so that the local truncation error of the method is atleast of 3rd order.

4. Use appropriate Adams-Bashforth formula, derived using the points $(x_n, y_n), (x_{n-1}, y_{n-1})$ and (x_{n-2}, y_{n-2}), to find $y(6)$ for **[5]**

$$y' = -4ty^2, \quad y(0) = 1$$

   with $h = 2$. Use implicit Euler's method to find any past value required and if needed one iteration of Newton-Raphson's method can be used with any initial value.

5. Given the characteristic polynomial $\rho(\eta) = \eta^2 - \eta - 1$, find the corresponding explicit linear multi-step scheme where kth order is given by **[5]**

$$\sum_{j=0}^{k} \alpha_j y_{n-j} = h \sum_{j=0}^{k} \beta_j f_{n-j}, \quad \beta_0 \in \mathbb{R}, \quad \beta_1 = 2, \beta_2 = -1, \quad \beta_{j=3,\ldots,k} = 0.$$

   Hence, determine the stability polynomial of the multi-step scheme obtained and check whether this polynomial is a Schur-polynomial or not.?

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