BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE PILANI
ME G535 Advanced Engineering Mathematics
Comprehensive Exam Part A

28^{th} December, 2022.

Instructions

- There are eight questions. This part is closed book. No calculators are allowed.
- Answer the questions in the given order. Start each question on a fresh page.

1. (5 marks) Find the eigen values of A, A^3, A^{-1}

A common blunt error in this question is converting matrix A into echelon form or upper triangular matrix.
PLEASE NOTE THAT Row transformations, preserve determinants but don't preserve Eigen values. The correct reasoning I've explained in the class and is also given in the textbook. Please understand the logic.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Solution: The determinant of an upper triangle matrix is the product of the diagonal terms. The eigen values are zeroes of the $|A - \lambda I| = (1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda)(5-\lambda)$. Hence eigen values of A are 1, 2, 3, 4 and 5.

$Ax = \lambda x$

$A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$

The eigen values of A are: $1, 2, 3, 4, 5$ 2 Marks

The eigen values of A^3 are: $1^3, 2^3, 3^3, 4^3, 5^3$ 2 Marks

Let

$$Ax = \lambda x \implies A^{-1}Ax = A^{-1}\lambda x \implies A^{-1}x = \frac{1}{\lambda}x$$

The eigen values of A^{-1} are: $1, 1/2, 1/3, 1/4, 1/5$ 2 Marks

Each sub-part will have 2 marks but maximum marks for the question is 5

2. (5 marks) The eigen values of a matrix A are given as 2 and 3 and the corresponding eigen vectors are [1, -2], and [2,1] respectively. Find the diagonalization of A and determine A.

Solution:

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \frac{1}{5} = \frac{1}{5} \begin{bmatrix} 14 & 2 \\ 2 & 11 \end{bmatrix}$$

Writing the matrix in the correct order is checked.
1 or 2 marks deducted for any miscalculation.

3. (5 marks) (a) Find the eigen values and eigen vectors of the rotation matrix $R(\theta)$.

$$R(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

3 Marks for eigen values and eigen vectors

Solution:
Eigen values of the matrix are $\cos\theta \pm i \sin\theta = e^{\pm i\theta}$, eigen vectors are [1,$\mp$ i]

(b) Find a third column and pre-multiplier α such that the matrix Q is orthonormal.

$$Q = \alpha \begin{bmatrix} 2 & -1 & \_ \\ -1 & 2 & \_ \\ 2 & 2 & \_ \end{bmatrix}$$

Solution: $\alpha = \frac{1}{3}$, the third column is [2, 2, -1]. This can be solved visually if you can, alternatively, you can use $Q^TQ = I$ for solving.

3 marks for this part. Maximum marks for Q3 is 5.

4. (5 marks) To study the absorption of nutrients in an insect gut we model its digestive tract by a tube of length l and cross-sectional area A. Nutrients of concentration μ=μ(x,t) flow through the tract at speed v, and are absorbed locally at a rate given by $A\mu^m$ (A and m are some constants). If the tract is empty at t=0, and then nutrients are introduced at the constant concentration μ_0 at mouth (x=0), derive the PDE and formulate the intitial boundary value problem.

Solution:

$$\mu_t + v\mu_x = -A\mu^m \quad \forall x \in [0, l]$$
$$\mu(x, 0) = 0 \quad \forall x \in [0, l]$$
$$\mu(0, t) = \mu_0 \quad \forall t > 0$$

Each line 2 marks
Max. 5 marks.

5. (6 marks) Find the Euler-Lagrange equations to determine the extremum of the following functionals

$$(i) J(y) = \int_0^1 (y'^2 + yy' + y^2)dx,$$
$$y(x) \in S = \{u | u \in C^2[0, 1], u(0) = 0, u(1) = 2\}$$
$$(ii) J(y) = \int_0^1 \left[\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} + y\right] dx$$
$$y(x) \in S = \{u | u \in C^5[0, 1], u(0) = u'(0) = u''(0) = u'''(0) = 0$$
$$u(1) = u'(1) = u''(1) = u'''(1) = 0\}$$
$$(iii) J(x, y) = \int_0^T [\ddot{x}y - \ddot{y}x]\, dt$$

Solution:
(a)

$$L(x, y, y') = y'^2 + yy' + y^2$$
$$L_y = y' + 2y$$
$$L_{y'} = 2y' + y$$
$$\frac{dL_{y'}}{dx} = 2y'' + y'$$
$$y'' - y = 0 \text{ (Euler equation)}$$

2 Marks

(b) Euler's equation

$$L_y - \frac{d}{dx}L_{y'} + \frac{d^2}{dx^2}L_{y''} - \frac{d^3}{dx^3}L_{y'''} + \frac{d^4}{dx^4}L_{y^{(iv)}} = 0$$
$$1 - 0 + 0 - 0 + 0 = 0$$

2 Marks

No solution exists.

(c) Euler equations are.

$$\ddot{x} - \ddot{x} = 0$$
$$\ddot{y} - \ddot{y} = 0$$

2 Marks

The equations are trivial and solution is flat.

6. (4 marks) Determine the natural boundary condition for the problem

$$J(y) = \int_0^1 (y'^2 + yy' + y^2)dx$$
$$y(x) \in S = \{u | u \in C^2[0,1], u(0) = 0\}$$

Solution: Natural boundary condition:

$$L_{y'} = 0, @x = 1$$
$$2y'(1) + y(1) = 0$$

4 Marks

7. (5 marks) Find the projection of b onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

Major errors in this are found to be writing matrix A as a single column.
Instead of inverting [ATA] dividing is done.
Order of matrix multiplication is important.
In correct order and illogical divisions are straight away given zero.
Since closed book, right formula may fetch 2 marks.
Only putting matrices in the right order is considered.
Final multiplication carries less weightage,

Solution:

$$Proj_A(b) = A(A^TA)^{-1}A^Tb$$

Substitute and simplify. Minor numerical errors are ignored.

8. (5 marks) If A=$\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$, find A^{100} by diagonalizing A.

Solution: The matrix is a Markov matrix and eigen values are 1 & 0.4 by inspection. Diagonalization of A

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1}$$
$$A^{100} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}^{100} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1}$$
$$= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Eigen values: 2 Marks
Eigen vectors: 2 Marks
Order and final steps FULL MARKS.