Mathematical Modeling

Math F420

Midterm Test

BITS Pilnai, K.K Birla Goa Campus

Date-28/09/2019
Duration: 90 minutes

75 Marks

- Answer all questions from the list in the answer sheet.
- For full credit please show all your work.

1. For a predator-prey systems:
$$\frac{du}{dv} = ug(u) - vp(u)$$
$$\frac{dv}{dt} = v(-d + q(u))$$
Let the functions p, q and g satisfy the following conditions.
   - $g(u) > 0$ for $u < K, g(u) < 0$ for $u > K, g(K) = 0$
   - $p(0) = 0, p(u) > 0$ for $u > 0$
   - $q(0) = 0, q^{'}(u) > 0$ for $u > 0$

   (a) Show that if a co-existence steady state exist,then there there are two saddle points $(0, 0)$ and $(K, 0.)$ In this case derive a condition for that co-existence steady state to be unstable.(8 M)

   (b) Sketch the typical phase-plane diagram in this case.(7 M)

2. Do the complete phase-plane analysis (with phase potrait) for the following non-linear system (15 Marks)
$$x^{''} + x^{'} + \epsilon x^2 = 0 : \epsilon < 0, \epsilon = 0, \epsilon > 0$$

3. Investigate the stability of the fixed point $(x, y) = (0, 0)$ for the system
$$x^{'} = \mu_1 x - y - \mu_2 x(x^2 + y^2) - x(x^2 + y^2)^2$$
$$y^{'} = x + \mu_1 y - \mu_2 y(x^2 + y^2) - y(x^2 + y^2)^2$$
describe and find any bifurcating families of periodic orbits that may exist. (10 M)

4. **Fish populations in a pond:** Imagine a small pond that is suitable enough to support two type of fish (say A, and B) who compete with each other for survival. Initially, assume that the pond environment can support an unlimited (exponetial growth) number of fish type A in isolation.

   (a) Write down an equation that describes the evolution of fish A population in absence of competition.(5 M)

   (b) Modify the equation to account for competetion of the type A with type B population for living space and a common food supply. You may assume that the growth rate of A population depends linearly on type B population.(5 M)

   (c) What are the steady states of the system ? Determine the stability of the steady states.(5 M)

5. Consider the nonliner equation for population growth

$$x_{n+1} = \frac{\lambda x_n}{1 + x_n}$$

   Where $\lambda > 0$

   Is there a non-trivial steady state ? If so find the stability of the steady state(s) and discuss any bifurcation that take place. (8)

6. Consider the logistic map $f(x) = rx(1-x)$ with $r > 0$. Find the range of r for which there is a stable 2-cycle for the system $x_{n+1} = f(x_n)$. (7 M)

7. Consider a system in polar form

$$r' = r - r^3$$
$$\theta' = 1 - \cos(2\theta)$$

   Find (describe) the euqilibrium points which are stable and also find the set of equilibrium points which are unstable. (5)