BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI (RAJASTHAN)

2022 - 2023 (SEMESTER I)

A

Classical Mechanics (PHY F211)

(Comprehensive Examination: Closed Book)

Date : 21st December 2022

Max. Marks: 120 [60 Closed book (Part I = 30 and Part II = 30) + 60 Open book]

Duration: 3 hrs [It is advisable to complete the closed book parts within maximum 1 hr 30 mins]

General instruction: Submit the Part I and II of the "closed book part" together. Question paper of the "open book part" will be supplied after the submission of Part I and II of the "closed book part".

Part I: Write the final answers only in the boxes provided back side of this page (Use only the last pages of the main answer copy for the rough work)

Q1. Consider a system of *five* point masses are moving in three-dimensional (3D) space. Two of them are connected by a rigid massless rod, and the remaining three masses are connected with each other also by massless rods. Find the number of generalized coordinates require to describe this system in 3D space.

Q2. A massless spring of rest length l_0 (with no tension) has a point mass m connected to one end and the other end fixed so the spring hangs under the gravitational field as shown in the figure at the bottom. Assume that the motion of the system is restricted on one vertical plane. Write down the Lagrangian in polar coordinates.

Q3. Find the equations of motion for a given time-dependent Lagrangian $L = \frac{1}{2}m\left(\dot{x}^2 - \omega^2 x^2\right)e^{2bt}$, where b is a constant.

Q4. Write down the Hamiltonian of a particle of mass m moving on the surface of a sphere of fixed radius R whose Lagrangian is: $L = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$, where θ and ϕ are the generalized coordinates of the system.

Q5. A particle of mass $m = 1$ moving in the field $V(x) = -2x$, travels from the point $x = 0$ to the point $x = 2$ in a time $t = 1$. Find the law of motion $x(t)$, which has a form $x(t) = At^2 + Bt + C$. **Clue:** Use the boundary conditions and then apply the least action principle.

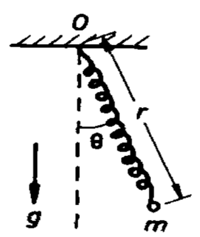
Q6. Determine the canonical transformation [i.e., write down $q = q(Q,P)$ and $p = p(Q,P)$] defined by the generating function: $F_1(q,Q,t) = \frac{1}{2}m\omega(t)q^2\cot Q$.

Q7. Write down the generating function $F_2(q,P,t)$ if the generating function $F_1(q,Q,t) = q(Q^2 + t^2)$.

Q8. Evaluate the Poisson brackets $\{A_1, A_2\}$ where $A_1 = \frac{1}{4}\left(x^2 + p_x^2 - y^2 - p_y^2\right)$ and $A_2 = \frac{1}{2}(xy + p_x p_y)$. All notations have usual meaning.

Q9. For the generating function $F_2(q,P,t) = e^{\gamma t}qP$, find the transformed Hamiltonian $K(Q,P,t)$ for an oscillator potential $V(q) = \frac{1}{2}\omega^2 q^2$. Here the mass $m = 1$ and γ is a constant.

Q10. Find the relation between the energy E and the action variable I for the one-dimensional linear Harmonic oscillator of mass m and angular frequency ω.



[3 × 10 = 30]

Answer sheet of Part I

(*Reminder: Use only the last pages of the main answer sheet for the rough work)

Name:

ID:

A1.

A2.

A3.

A4.

A5.

A6.

A7.

A8.

A9.

A10.

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Part II

Instruction: Question paper of the "open book part" will be supplied after the submission of the main answer copy of the "closed book part" and the Part II.

Q11. A smooth wire has the form of the helix $x = a\cos\theta, y = a\sin\theta, z = b\theta$, where θ is a real variable, and a, b are positive constants. The wire is fixed with a point on the z-axis pointing vertically upwards. A particle of mass m can slide freely on the wire.

(a) Construct the Lagrangian in terms of the generalized coordinate θ.

(b) Derive the Hamiltonian.

(c) Construct the Hamilton's equation of motion.

(d) Derive the equation of motion.

(e) Derive the trajectory $\theta(t)$ for the initial condition, $z = 0$ and $v_z = v_0$ when $t = 0$.

[3+7+2+6=18]

Q12. A particle of mass m moves in periodic motion in one dimension under the influence of a potential $V(x) = -k/|x|$, where F is a constant. For energies that are negative the motion is bounded and oscillatory. By the action-angle variables find the period of the motion as a function of the particle's energy.

[12]

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Duration : 3 hrs – (Time taken in the "Closed Book" part)

1. Consider the Hamiltonian of a linear harmonic oscillator of mass $m = 1$, which is perturbed by an anharmonic potential. The Hamiltonian of this system is
$$H = \frac{1}{2}(p^2 + \omega^2 x^2) + \alpha x^3 + \beta x p^2.$$
Here the anharmonic part is perturbative under the assumption that $|\alpha x| \ll \omega^2, |\beta x| \ll 1$.

   (a) Consider the generating function $F_2(x,P) = xP + ax^2P + bP^3$. Determine the parameters a and b such that the new Hamiltonian $K(Q,P)$ does not contain any anharmonic terms up to first-order terms in $\alpha\omega^{-2}Q$ and βQ.

   Clue: Follow the steps given below.

   - *Step 1*: Find the canonical transformation $x = x(Q,P)$ and $p = p(Q,P)$. However, it is cumbersome to derive the exact form of $x(Q,P)$ and $p(Q,P)$. One can exploit the perturbative nature of the anharmonic potential and calculate $x(Q,P)$ and $p(Q,P)$ approximately. Here, the first term of the generating function F_2 is xP, which represents an identity transformation ($p = P, Q = q$). The other two terms are considered due to the perturbative anharmonic part in the Hamiltonian.
   - *Step 2*: Derive $p = p(x,P), Q = Q(x,P)$ from F_2. Due to the reason mentioned in Step 1 the terms appeared in the expressions of $p = p(x,P), Q = Q(x,P)$, other than the identity transformation ($p = P, Q = q$), must be very small. The trick is to substitute $x = Q$ and $p = P$ in the smaller terms of $p = p(x,P), Q = Q(x,P)$ and approximately obtain the canonical transformation.
   - *Step 3*: Construct the new Hamiltonian $K(Q,P)$ neglecting all the terms of fourth degree in Q and P (i.e., neglect $Q^4, QP^3, Q^2P^2, Q^3P, P^4$).
   - *Step 4*: Determine the parameters a and b to drop the remaining anharmonic terms.

   (b) Determine $x(t)$.

[(4+2+12+6)+6=30]

2. A point charge of mass $m = 1$ is moving under the influence of a central potential and an electric field applied along z-direction. The potential of this system is
$$V(r,z) = -\frac{k}{r} - Fz.$$
The Hamilton's principal function of this system can be represented in a complete separable form in the parabolic coordinates (ξ, η, ϕ), where these coordinates are related to the Cartesian coordinates as
$$x = \sqrt{\xi\eta}\cos\phi,\ y = \sqrt{\xi\eta}\sin\phi,\ z = \frac{1}{2}(\xi - \eta).$$
Also note that $\rho \equiv \sqrt{x^2 + y^2} = \sqrt{\xi\eta}$ and $r = \sqrt{\rho^2 + z^2} = \frac{1}{2}(\xi + \eta)$.

   (a) Derive the kinetic energy $T = \frac{1}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$ in terms of the parabolic coordinates (ξ, η, ϕ) and then construct the Lagrangian.

   (b) Construct the Hamiltonian using Legendre transformation.

   (c) Besides energy, identify another constant of motion. (**Clue:** There are *two* constants of motion)

   (d) Assume the separable form of the Hamiltons' principal function
   $$S = S_\xi(\xi, \lambda_1, \lambda_2) + S_\eta(\eta, \lambda_1, \lambda_2) + S_\phi(\phi, \lambda_1, \lambda_2) - Et,$$
   where λ_1 and λ_2 are two constants of motion. Set up the Hamilton-Jacobi equations.

   (e) Write down the Hamiltons' principal function $S(\xi, \eta, \phi, \lambda_1, \lambda_2, \lambda_3)$ in terms of the integrals (no need to integrate). Here, λ_3 is another constant, which has appeared while separating the variables.

[7+7+2+7+7=30]