BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI (RAJASTHAN)

2023 - 2024 (SEMESTER I)

Classical Mechanics (PHY F211)

Mid-Semester Examination (Closed Book)

14th October 2023

Max. Marks : 90 **Duration : 1 hr 30 mins**

Please check that both sides of this question paper are printed

1. Choose the most appropriate generalized coordinate(s) of the following systems and construct their Lagrangian:

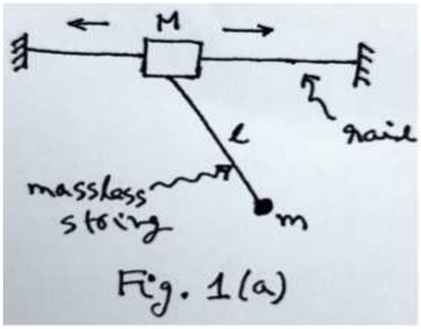
   (a) A mass M is free to slide along a frictionless rail, which is fixed between two walls. A pendulum of length l and mass m hangs from the mass M [**shown in Fig. 1(a)**].

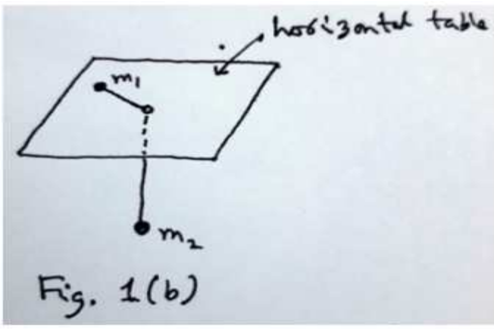
   (b) Two mass points m_1 and m_2 $(m_1 \neq m_2)$ are connected by a massless string of length l passing through a hole in a horizontal table [**shown in Fig. 1(b)**]. The string and the mass m_1 move without friction on the table, and m_2 free to move in a vertical line.

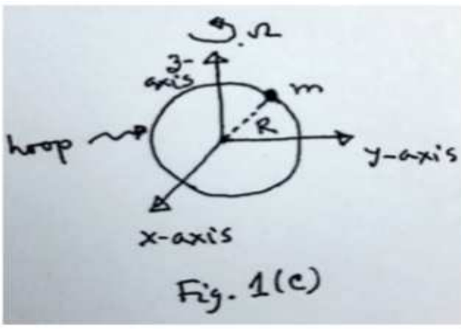
   (c) A bead of mass m is constrained to move on a hoop of radius R. The hoop rotates with constant angular velocity Ω around a vertical axis which coincides with a diameter of the hoop [**shown in Fig. 1(c)**]. [**Hint:** Use spherical polar coordinates (r, θ, ϕ), and you may assume that at $t = 0$, $\phi = 0$]

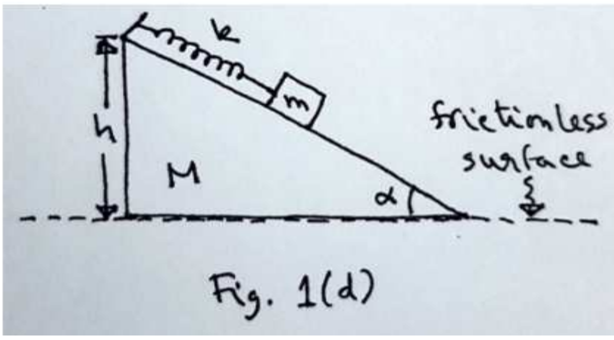
   (d) A block of mass m is attached to a wedge of mass M and height h by a spring with spring constant k [**shown in Fig. 1(d)**]. The angle of the inclined frictionless surface of the wedge with respect to the horizontal *frictionless surface* is α. Assume, the relaxed length of the spring is d.

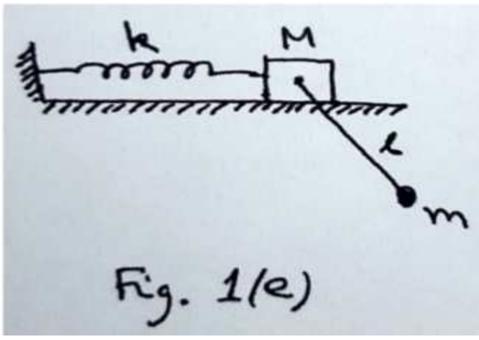
   (e) A simple pendulum consisting of mass m and weightless string of length l is mounted on a support of mass M which is attached to a horizontal spring with spring constant k [**shown in Fig. 1(e)**]. The mass M can move horizontally without any friction.











[5×5=25]

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2. (a) i. Using Euler-Lagrange's equation, derive the equation(s) of motion of the system described in **Fig. 1(a)**.
   ii. For this system, find the general solution of the equation(s) of motion for small oscillations, i.e., assume $\sin\theta = \theta$ and $\cos\theta = 1 - \frac{1}{2}\theta^2$.

   (b) i. Using Euler-Lagrange's equation, derive the equation(s) of motion for the system described in **Fig. 1(c)**.
   ii. Find the equation of the small oscillation motion of the mass about the stable equilibrium point $\theta = \pi$, i.e., at the bottom of the hoop. [**Hint:** Substitute $\theta = \pi + \alpha$ in the equation of motion, where α is very small, and find the equation of motion for the variable α]
   iii. Find the critical angular velocity Ω_c such that $\theta = \pi$ remains the stable equilibrium point.

[(5+9)+(8+5+3)=30]

3. A particle is constrained to move on a plane. The particle is attracted to a fixed point P in this plane, i.e., the force is always directed exactly at P. The force law is given as

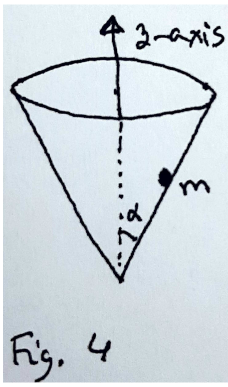
$$\vec{F} = \left(-\frac{k_1}{r^2} + \frac{k_2}{r^3}\right)\hat{r},$$

where k_1 and k_2 are constants, and assume $k_1 > 0$. The variable r is the distance of the particle from the point P and $\hat{r}$ is the unit vector along r-direction.

   (a) Construct the Lagrangian of this system in the most appropriate coordinates, i.e. in polar coordinates (r, θ), and their velocities.
   (b) Find the equations of motion and show that the angular momentum L is conserved.
   (c) Assume $L^2 > mk_2$. Find the equation for the orbit, i.e., r as a function of θ. [**Hint:** Substitute $r = 1/u$ in the equations of motion. Construct a differential equation with u as a function of θ and solve the equation]

[6+5+7=18]

4. A particle of mass m moves on the inner frictionless surface of a vertical cone, symmetric about the z-axis and having equation $x^2 + y^2 = z^2 \tan\alpha^2$ [**shown in Fig. 4**].



   (a) *Using the above constraint*, construct the Hamiltonian of the system starting from its Lagrangian and using Legendre transformation. Here, the height of the cone is sufficiently large, and therefore, the motion of the mass m is restricted on the cone's inner surface. [**Hint:** Since the system has the *cylindrical symmetry*, the cylindrical coordinates (r, θ, z) are appropriate for constructing the Hamiltonian. Note that, in the case of the cylindrical coordinate system, the coordinates (r, θ) are identical to the polar coordinates in the $x - y$ plane, and z is identical to the Cartesian z-coordinate]
   (b) Derive the equations of motion of this system using Hamilton's equations.

[10+7=17]