BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI (RAJASTHAN)

2023 - 2024 (SEMESTER I)

Classical Mechanics (PHY F211)

(Comprehensive Examination: Closed Book)

Date : 20th December 2023

Max. Marks: 120 [60 Closed book + 60 Open book]

Duration: 3 hrs [It is advisable to complete the closed book parts within maximum 1 hr 30 mins]

General instruction: Question paper of the "open book part" will be supplied after the submission of the "closed book part".

Write the final answers only in the boxes provided separately

(Use only the last pages of the main answer copy for the rough work)

Q1. A bead of mass m slides without friction along a wire which has the shape of a parabola $y = Ax^2$ (A is a constant) with axis vertical in the earth's gravitational field with acceleration due to gravity g. Consider the horizontal displacement x as the generalized coordinate and write down the Lagrange's equation of motion.

Q2. The point of support of a simple plane pendulum moves vertically according to $y = h(t)$, where $h(t)$ is some given function of time. Construct the Lagrangian considering the angle θ the pendulum makes with with the vertical as the generalized coordinate.

Q3. A particle moves vertically in the uniform gravitational field with the acceleration due to gravity g near the surface of the earth. The Lagrangian for this case is $L = \frac{1}{2}m\dot{z}^2 - mgz$. Suppose at the initial time $t = 0$, the particle is at $z = 0$, and at a later time $t > 0$ it is at z. For any motion between these points the action is $S[z(t)] = \int_0^t L(z,\dot{z})\,dt$. Suppose by some means we know that the path followed by the particle is $z(t) = z_0 + v_0 t + \frac{1}{2}at^2$, where z_0 and v_0 are chosen that $z(t)$ passes through the end points, and a is just a parameter which can be adjusted. Calculate $S[z(t)]$ for the above path.

Q4. A system with one degree of freedom has a Hamiltonian $H(p,q) = \frac{p^2}{2m} + f(q)p + g(q)$. Here all the notations have their usual meaning. Find the Lagrangian $L(q,\dot{q})$.

Q5. Using the method of action-angle variables, evaluate the time period $T(E)$ for a bouncing ball, i.e., $V(x) = mgx$ (for $x > 0$) and $V(x) = \infty$ (for $x \leq 0$).

Q6. Consider 2D motion of a particle of mass m in an isotropic harmonic oscillator potential $V(r) = \frac{1}{2}kr^2$. Calculate the time-derivative of a dynamical variable $f(p_x, p_y, x, y) = \frac{1}{m}p_x p_y + kxy$.

Q7. A particle moving vertically in a uniform gravitational field with the acceleration due to gravity is g. Write down the Hamiltonian of this system in terms of a new set of canonical variable $Q = -p_z$, $P = z + Ap^2$, where A is adjustable parameter, z is the vertical direction, and p_z is the canonical momentum conjugate to z.

Q8. Find the conditions on the small parameters a, b, c, d, e, and f in order that $q = Q + aQ^2 + 2bQP + cP^2$, $p = P + dQ^2 + 2eQP + fP^2$ be a canonical transformation to first order (ignore higher order terms of the small parameters) in small parameters.

Q9. The transformation equations between two sets of coordinates are $Q = \ln(1 + \sqrt{q}\cos p)$, $P = 2\sqrt{q}(1 + \sqrt{q}\cos p)\sin p$. Find the relations $q = q(Q,P)$ and $p = p(Q,P)$.

Q10. For the generating function $F_2(q,P,t) = e^{\gamma t}q^2 P$, find the transformed Hamiltonian $K(Q,P,t)$ for an oscillator potential $V(q) = \frac{1}{2}\omega^2 q^2$. Here the mass $m = 1$ and γ is a constant.

Q11. Write down the Euler-Lagrange's equations for the Lagrangian of the couple quartic oscillator $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k(x^4 + y^4) + \alpha xy$, where m, k and α are constants. All the dynamical variables have their usual meaning.

Q12. Derive the Hamilton's principal function $S(q,t,\alpha)$ for the free particle (i.e., the Hamiltonian $H = \frac{1}{2}p^2$), where α's are constants of motion.

[5 × 12 = 60]

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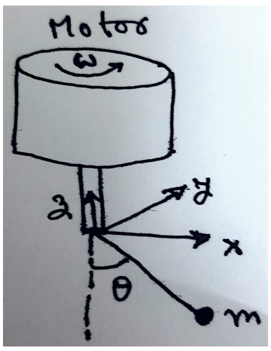
1. Consider a motion of a particle in a field with non-central potential $V = k\cos\theta/r^2$, where k is a constant.
   (a) Write down the time-independent Hamilton-Jacobi equation for the generating function S_0 in spherical polar coordinates. Note that the form of the kinetic energy in the spherical polar coordinate was discussed in the class.
   (b) Solve the Hamilton-Jacobi equation by the method of separation of variables, derive the expression for S_0 of the form $S_0(r,\theta,\phi;E,\alpha_2,\alpha_3)$, where E is the total energy, and (α_2,α_3) are two more separation constants. Note that, your answer has certain integrals, but you do not need to evaluate these at this stage.
   (c) Interpret physically the separation constants (α_2,α_3) by obtaining p_r, p_θ, p_ϕ in terms of $(r,\theta,\phi,E,\alpha_2,\alpha_3)$.
   (d) From (c), show that the z-component of the angular momentum L_z (which is also the same as p_ϕ) is constant.
   (e) From (c), also show that $L^2 + 2mk\cos\theta$ is also a constant (and equals to a separation constant), where $L^2 = p_\theta^2 + p_\phi^2/\sin^2\theta$ is the square of the total angular momentum of the particle.
   (f) Consider the one of the other half of the transformation equations $\partial S_0/\partial E = t + \beta_E$, find the radial coordinate r as a function of time.

[4+11+5+5+5=30]

2. A motor rotates a vertical shaft (just below the motor) which is attached to a simple pendulum of length l and mass m as shown in the figure. The pendulum is constrained to move in a plane, and this plane is rotated at constant angular velocity ω by the motor.
   (a) Construct the Lagrangian of the mass m, where θ will be the generalized coordinate. Remember to consider an additional velocity of m due to the rotation of the plane of oscillation ($x-y$ plane) about z-axis with angular velocity ω.
   (b) Calculate the equation of motion using the Euler-Lagrange equation.
   (c) Find the two equilibrium positions θ_1 and θ_2 of the mass m by setting $\ddot{\theta} = 0$.
   (d) Find the equation of motion of the mass m about the each equilibrium position. Assume, the standard small oscillation approximation about the equilibrium points.
   (e) Find the angular velocities ω_1 and ω_2 of the small oscillations respectively about the two equilibrium positions (θ_1,θ_2).
   (f) Derive the angular momentum L $(= I\omega)$ of the mass m about the z-axis, where I is the moment of inertia of the point mass m about the z-axis.
   (g) Calculate the torque $\frac{dL}{dt}$ the motor must supply to maintain the small oscillation about each of the equilibrium positions.



[8+4+3+6+3+3+3=30]