Mid-Semester Examination

Quantum Mechanics II (PHY F311)

Date : 05.11.2022 **Maximum Marks : 90** **Time : 90 minutes**

Q1a) Let $\hat{A}$ be a Hermitian operator and $\hat{B}$ an anti-Hermitian operator. What can you say about the nature (Hermitian, anti-Hermitian, neither) of i) $[\hat{A},\hat{B}]$ ii) $\{\hat{A},\hat{B}\}$?

b) Prove that the expectation value of an observable is guaranteed to be positive if and only if all its eigenvalues are positive.

c) Prove that the time evolution operator for a time-independent Hamiltonian can be expressed as

$$\sum_n e^{-iE_n t/\hbar} |\phi_n><\phi_n|$$

where E_n & $|\phi_n>$ are the energy eigenvalues and the corresponding energy eigenvectors.

d) Let U & U' be two disjoint subspaces (the only common element is the null vector) of a vector space, that are not mutually orthogonal. Let $\hat{P}_U$ & $\hat{P}_{U'}$ be the orthogonal projection operators on these subspaces. Do $\hat{P}_U$ & $\hat{P}_{U'}$ commute? Will they commute if U & U' are mutually orthogonal?

e) Prove that the z-component of the angular momentum, $\hat{L}_z$, of a particle moving in a potential that depends on the z coordinate alone $\left(\text{i.e.,} \hat{H} = \vec{P}^2/2m + V(z)\right)$ is a constant of motion. (7 x 5)

Q2) Calculate the uncertainties δS_x & δS_y of the spin operators $\hat{S}_x$ & $\hat{S}_y$ in the state $|+>$ and show that they satisfy the generalized uncertainty relation. (10)

Q3) Consider a 3-states quantum system with a state vector

$$|\psi> = |\phi_1> + i|\phi_2> - i|\phi_3>$$

where $|\phi_1>, |\phi_2>$ & $|\phi_3>$ are eigenvectors of an observable $\hat{A}$ with eigenvalues a, 2a and a respectively. These vectors form an orthonormal basis.

a) If the value 'a' is obtained after the measurement of the observable $\hat{A}$, then what will be the state after the measurement?

b) Consider another observable $\hat{B}$ given as

$$\hat{B} = b|\phi_1><\phi_2| + b|\phi_2><\phi_1| + b|\phi_3><\phi_3|$$

If this observable is measured on the system after the measurement of $\hat{A}$ has resulted in the value 'a', then

i) What are the possible values one will get as the outcome of the measurement ii) what are their probabilities

c) If the measured value of $\hat{B}$ in part (b) is ‘b’, then what is the state ket after this measurement?

d) If now, observable $\hat{A}$ is measured again on the state after the measurement of $\hat{B}$ has resulted in the value ‘b’, what is the probability of getting the earlier measured value ‘a’? Hence, what can you say about the compatibility of the two observables? (3 + (8 + 4) + 3 + 7)

Q4) The quantum virial for a particle, which is an operator, is defined as

$$\hat{G} = \hat{\vec{X}} \cdot \hat{\vec{P}}$$

a) Derive the Heisenberg equation of motion for the virial as

$$\frac{d\hat{G}}{dt} = 2\hat{T} - \hat{\vec{X}} \cdot \vec{\nabla}\hat{V}$$

b) Using the above equation, show that in an energy eigenstate, and in Coulomb potential $V(\vec{x}) = -\frac{k}{r}$ $(r = |\vec{x}|)$, the expectation value of the KE is negative half of the expectation value of the potential energy:

$$< T > = -\frac{< V >}{2}$$

(15 + 5)

Or

a) Obtain the Heisenberg equations of motion of a charged particle moving in a magnetic field (no electric field). (15)

b) Under what conditions, will the expectation value of position describe the classical trajectory? (5)

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Useful formulas:

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; |s_x;\pm> = \frac{1}{\sqrt{2}}[|+> \pm |->] ; |s_y;\pm> = \frac{1}{\sqrt{2}}[|+> \pm i|->]$$

$$\frac{d\hat{A}_H}{dt} = \frac{1}{i\hbar}[\hat{A}_H, \hat{H}] ; \hat{H}_{em} = \frac{1}{2m}\left(\vec{P} - q\vec{A}\right)^2 + qV ; \hat{U}(t) = e^{-i\hat{H}t/\hbar}$$

$$(\delta A)^2 (\delta B)^2 \geq \frac{1}{4}\left|<[\hat{A}, \hat{B}]>\right|^2$$