Comprehensive Examination

Quantum Mechanics II (PHY F311)

Date : 30.12.2022 **Maximum Marks : 120** **Time : 3 Hours**

Closed-Book (50 Marks)

Q1) Show that the harmonic oscillator creation operator a^+ does not have any eigenvectors. That is, show that the only vector $|\lambda>$ that satisfies the eigenvalue equation

$$a^+ |\lambda> = \lambda|\lambda>$$

is the null vector. (8)

Q2) Consider a spin-$1/2$ particle confined in a rigid-one dimensional box of length ℓ (0 to ℓ). The normalized wave function of the particle is given by

$$\sqrt{\frac{2}{\ell}}\begin{pmatrix}\frac{1}{2}\sin\frac{\pi x}{\ell} \\ \frac{\sqrt{3}}{2}\sin\frac{2\pi x}{\ell}\end{pmatrix}$$

a) What is the probability of finding the particle with spin down?

b) What is the mean value of the energy of the particle? (4 + 4)

Q3) Obtain the matrix representation of $\hat{S}_y$ for a spin-1 particle in the eigenbasis of $\hat{S}_z$. Hence find the eigenvector of $\hat{S}_y$ with eigenvalue -1, i.e., find $|1,-1>_y$. (7 + 3)

Q4) Obtain the angular representation

$$Y_\ell^{-\ell}(\theta,\phi) = <\theta,\phi|\ell,-\ell>$$

by using the fact that $\hat{L}_-|\ell,-\ell> = 0$. You may not normalize $Y_\ell^{-\ell}(\theta,\phi)$. (8)

Q5) Show that the ket

$$|j_1,j_2;j_1,j_2> = |j_1,j_1> \otimes |j_2,j_2>$$

from the simultaneous eigenbasis of the commuting operators $\{\vec{J}_1^2, \vec{J}_2^2, J_{1z}, J_{2z}\}$ is an eigenvector of $\vec{J}^2$, where $\vec{J} = \vec{J}_1 + \vec{J}_2$. What is the eigenvalue? (8)

Q6) Calculate the first order correction to the energy of the ground state of the Harmonic oscillator, when an an-harmonic potential energy term $\alpha x^3 + \beta x^4$ acts as a perturbation to the harmonic potential $\frac{1}{2}m\omega^2 x^2$. (8)

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Useful Formulas :

$$J_{\pm} \mid j,m> = \hbar\sqrt{j(j+1)-m(m\pm 1)} \mid j,m\pm 1>$$

$$\hat{L}_{\pm} = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right)$$

Open Book (70 Marks)

Q1a) Show that

$$\int_{-\infty}^{\infty} dx \, | x+a >< x+a | = I$$

Using the above, show that $\hat{T}(a)\hat{T}^{+}(a) = I$. (2 + 4)

b) Let a particle have a potential energy function $V(\vec{x})$ that has an absolute minima V_{min}, i.e., $V(\vec{x}) \geq V_{min}$. Show that the expectation value of potential energy cannot be less than V_{min} in any state, i.e., $< \psi | V | \psi > \geq V_{min}$. (7)

c) Calculate $\hat{x}(t) = U^{+}(t)\,\hat{x}(0)\,U(t)$ for a free particle by directly calculating the RHS. (7)

d) Find the uncertainty of energy in a general coherent state $|\lambda>$ of the one-dimensional harmonic oscillator. (7)

e) Prove that the Virial operator $\hat{G} = \vec{x} \cdot \vec{p}$ commutes with all three components of the angular momentum operator $\vec{L}$. Hence show that the operator remains invariant under rotation, i.e., it is a scalar operator. (7)

[If you are loath to provide a general proof, you may calculate the commutation with only one of the components, L_x ,say, and generalize]

f) If you add two spins angular momenta, $\vec{S}_1$ & $\vec{S}_2$, the first one of spin 1 and the second of spin ½, then

i) express the ket $|1,1/2;3/2,3/2>$ from the $\{|s_1,s_2;s,m>\}$ basis in terms of the $\{|s_1,s_2;m_1,m_2>\}$ basis.

ii) Which kets from the $\{|s_1,s_2;m_1,m_2>\}$ basis will occur in the expansion of the ket $|1,1/2;1/2,-1/2>$? No need to calculate the expansion. (3 + 4)

g) Consider a one-dimensional harmonic oscillator. Let a perturbative Hamiltonian of the following form

$$\hat{H}_1 = \varepsilon \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[|0><n| + |n><0| \right]$$

be added to the unperturbed Hamiltonian. Calculate the first non-zero correction to the energy of the ground state. (7)

[You may use the summation formula : $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$]

h) Consider a particle moving on the surface of a sphere of radius R. The Hamiltonian, which is only the kinetic energy, is given by

$$\hat{H} = \frac{\hat{\vec{L}}^2}{2mR^2}$$

Now, a perturbative potential energy, of the form $\omega \hat{L}_z$, is added to the Hamiltonian (for example, if the particle has a charge and a magnetic field $B\hat{z}$ is applied).

Obtain the Heisenberg equations of motion for the three components of angular momentum. Is any component of angular momentum conserved? (6)

Q2) Consider a two-states system with a Hamiltonian

$$\hat{H} = E_0\left(|1><1| + |2><2|\right) + \delta\left(|1><2| + |2><1|\right)$$

a) Calculate the energy eigenvalues and eigenvectors. (6)

b) If $\delta << E_0$, the first bracket on the RHS above can be termed as $\hat{H}_0$ and the second bracket as $\hat{H}_1$ (the constants are taken with the brackets), so that $\hat{H}_0$ is the unperturbed Hamiltonian and $\hat{H}_1$ is the perturbation. Now apply first order degenerate perturbation theory to the eigenvalues of $\hat{H}_0$ to obtain i) first order-corrected energy levels and ii) the correct zeroth order eigenvectors when the perturbation is removed. Compare these to those of the full Hamiltonian calculated in part (a). (10)

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