Birla Institute of Technology and Science, Pilani
Mid-Semester Examination
Quantum Mechanics II (PHY F311)

Date: 09.10.2023 **Time: 90 Minutes**

Q1a) Prove that an orthonormal set of vectors, $|\phi_1>, |\phi_2>, ..., |\phi_n>$, is linearly independent.

b) Prove that a unitary operator $\hat{U}$ maps an orthonormal set of vectors to another orthonormal set of vectors. Hence show that a unitary operator maps an orthonormal basis to another orthonormal basis.

c) Let $\hat{A}$ be a Hermitian operator. Prove that the expectation value of $\hat{A}^2$ in any arbitrary state $|\psi>$ is positive, i.e.,

$$<\psi|\hat{A}^2|\psi> \geq 0$$

What can you say about the same expectation value if $\hat{A}$ were an anti-Hermitian operator? (6+8+6)

Q2a) Prove that if two operators have a complete set of simultaneous eigenvectors, then they will commute.

b) Let $\hat{A}, \hat{B} \& \hat{C}$ be three operators such that both $\hat{B} \& \hat{C}$ commute with $\hat{A}$ but they do not commute with each other. Then prove that $\hat{A}$ must necessarily have degenerate eigenvalues. (6+8)

[**Hint**: Assume to the contrary that all the eigenvalues of $\hat{A}$ are non-degenerate and arrive at a contradiction in the light of the statement of part (a)]

Q3a) Using the commutation relation of the three spin operators $\hat{S}_x, \hat{S}_y \& \hat{S}_z$, calculate the following commutator

$$[\vec{S}\cdot\hat{n}, \vec{S}\cdot\hat{m}] \qquad (\hat{n} \;\&\; \hat{m} \text{ are unit vectors})$$

and show that it is of the form $i\hbar\vec{S}\cdot\hat{p}$ where $\hat{p} = \hat{n}\times\hat{m}$.

[You may use the compact commutatitor formula $[\hat{S}_i, \hat{S}_j] = i\hbar\varepsilon_{ijk}\hat{S}_k$] (8)

In the following part , $\hat{S}_n \;\&\; \hat{S}_m$ are short-hand notations for $\vec{S}\cdot n \;\&\; \vec{S}\cdot\hat{m}$ respectively.

b) In view of the result of part (a) and the generalized uncertainty relation, what can you say about the lower bound for the uncertainty product $\Delta S_n \cdot \Delta S_m$ in the states $|\hat{n};\pm>$ or $|\hat{m};\pm>$? Show that the uncertainty product $\Delta S_n \cdot \Delta S_m$ in these states is actually equal to the lower bound predicted from the uncertainty relation. You need not do any detailed calculations in answering the above. (12)

Q4) Let $\hat{T}(\delta x)$ be the infinitesimal translation operator along the x-axis (1-d motion).

a) Calculate $\hat{T}^{+}(\delta x)\hat{X}\hat{T}(\delta x)$ & $\hat{T}^{+}(\delta x)\hat{P}\hat{T}(\delta x)$ to first order in the infinitesimal quantity δx. (8)

b) Using the result of part (a) calculate the change in the expectation values of both position and momentum as a result of this infinitesimal translation of a state vector $|\psi>$. (8)

Q5) Consider a three-states system for which an observable $\hat{A}$ has eigenvalues a, a, and 2a with corresponding orthonormal eigenvectors $|\phi_1>, |\phi_2> \& |\phi_3>$. The Hamiltonian of the system in this eigenbasis of $\hat{A}$ is given by

$$\hat{H} = b|\phi_1><\phi_1| + b|\phi_2><\phi_3| + b|\phi_3><\phi_2| \quad \text{(a and b are real constants)}$$

At t = 0, the state of the system is given by

$$|\psi(0)> = |\phi_2>$$

a) Find the state at time t, i.e., $|\psi(t)>$ in terms of the eigenbasis of $\hat{A}$. (12)

b) Find the expectation value of the observable $\hat{A}$ both at t = 0 and at time t. (8)

[**Hint**: You need the energy eigenbasis for calculating time evolution.]

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