Comprehensive Examination

Quantum Mechanics II (PHY F311)

Date : 07.12.2023 **Maximum Marks : 120** **Time : 3 Hours**

Closed Book

Q1. Prove that if two operators $\hat{A}$ & $\hat{B}$ commute, then an eigen-subspace of one will be invariant (closed) under the action of the other. What is its implication when the eigenvector of one of the two operators is non-degenerate? (8)

Q2. Prove that all the eigenvalues of a unitary operator are of unit absolute value and eigenvectors corresponding to distinct eigenvalues are mutually orthogonal. (8)

Q3. Calculate the commutator

$$[\hat{\pi}_x, \hat{\pi}_y]$$

and relate it to the magnetic field operator. Here, $\vec{\pi}$ is the kinetic momentum operator of a charged particle in an electromagnetic field. (8)

Q4. Obtain the time-evolution of a coherent state and show that

$$|\alpha; t> = |\alpha e^{-i\omega t}> \qquad (8)$$

Q5. By using the condition

$$\hat{L}_- Y_\ell^{-\ell}(\theta,\phi) = 0$$

and using the ϕ dependence of the spherical harmonics, determine $Y_\ell^{-\ell}(\theta,\phi)$. You need not determine the normalization constant. (8)

Q6. Calculate the second order correction to the energy of the n^{th} energy state $|n>$ of a one-dimensional harmonic oscillator due to a perturbing potential term αx^3. (10)

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Relevant formulas:

1. $\vec{\pi} = \vec{p} - q\vec{A}$

2. $\hat{L}_- = i\hbar e^{-i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta \frac{\partial}{\partial\phi} \right)$

Open Book

Q1) Consider a two-dimensional anharmonic oscillator with the Hamiltonian

$$\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}m\omega^2(\hat{x}^2 + \hat{y}^2) + \delta m\omega^2\hat{x}\hat{y}$$

Now, consider the following operators

$$\hat{X} = \frac{1}{\sqrt{2}}(\hat{x}+\hat{y});\ \hat{Y} = \frac{1}{\sqrt{2}}(\hat{x}-\hat{y});\ \hat{P}_x = \frac{1}{\sqrt{2}}(\hat{p}_x + \hat{p}_y);\ \hat{P}_y = \frac{1}{\sqrt{2}}(\hat{p}_x - \hat{p}_y)$$

a) Using Heisenberg equations of motion for these operators, show that

$$\text{i)}\ \frac{d\hat{X}}{dt} = \frac{\hat{P}_x}{m}\ ;\ \frac{d\hat{P}_x}{dt} = -m\omega_x^2\hat{X} \qquad (\omega_x = \omega\sqrt{1+\delta})$$

(20)

$$\text{ii)}\ \frac{d\hat{Y}}{dt} = \frac{\hat{P}_y}{m}\ :\ \frac{d\hat{P}_y}{dt} = -m\omega_y^2\hat{Y} \qquad (\omega_y = \omega\sqrt{1-\delta})$$

b) How would you interpret the above equations of motion. (5)

Q2a) What are the possible values one would get if J_x & J_y are measured in the state $|\, j\ m >$, which is an eigenstate of J^2 & J_z? Answer with adequate explanation. (5)

b) What are the expectation values $< J_x >$ & $< J_y >$ in the state $|\, j\ m >$? (5)

c) Express $\left(J_x^2 + J_y^2\right)$ in terms of J_+ & J_- and hence calculate $\left(J_x^2 + J_y^2\right)|\, j\ m >$. Using the result, calculate $\left(J_x^2 + J_y^2 + J_z^2\right)|\, j\ m >$, and explain the result you get. (6)

d) Using the commutators

$$[\hat{L}_z, \hat{P}_x] = i\hbar\hat{P}_y\ \&\ [\hat{L}_z, \hat{P}_y] = -i\hbar\hat{P}_x$$

show that

$$e^{\frac{i}{\hbar}\phi\hat{L}_z}\hat{P}_x e^{-\frac{i}{\hbar}\phi\hat{L}_z} = \cos\phi\,\hat{P}_x - \sin\phi\,\hat{P}_y$$

$$e^{\frac{i}{\hbar}\phi\hat{L}_z}\hat{P}_x e^{-\frac{i}{\hbar}\phi\hat{L}_z} = \sin\phi\,\hat{P}_x + \cos\phi\,\hat{P}_y$$

(Use Hausdorff-Baker-Campbell formula and get the pattern after a few terms in the series) (12)

Q3) Consider a two-dimensional isotropic harmonic oscillator with a small coupling term that acts as the perturbation. The complete Hamiltonian is given by

$$\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}m\omega^2(\hat{x}^2 + \hat{y}^2) + \delta\hat{x}^3\hat{y}$$

a) Calculate the first order splitting of the second energy level. (14)

b) Will there be a first order shift in the first energy level? (3)

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