BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE – PILANI, K.K.B.G.C
FIRST SEMESTER 2022-2023
STATISTICAL MECHANICS - COMPREHENSIVE EXAMINATION

PHY F312 | DECEMBER 31, 2022
TOTAL MARKS: 75 | DURATION: 3 HOURS

NAME: ID NO.:

Instructions: (i) *Part A is to be answered in the space provided after each question. Only the key steps needs to be given.* (ii) *Use the back of main answer sheet for rough work for Part A* (iii) *Part B is to be answered in the main answer sheet.*

Part A (*Each question carries 5 marks*)

1. A *non-ideal* gas undergoes adiabatic, free expansion. Which of the following remain(remains) constant in the process and which one(ones) does(do) not: (i) entropy, (ii) internal energy and (iii) temperature. Give a brief justification for your choices.

2. N non-interacting spin-half fermions are placed in a $1D$ harmonic potential with single particle energy levels given by $(n + \frac{1}{2})\hbar\omega$. Find the Fermi energy for the system. Assume N to be even.

3. Hamiltonian for a system of particles in $1D$ is given to be $\sum_i (p_i^2/2m + \alpha x_i^4)$. Use equipartition theorem $\left(\left\langle y_j \frac{\partial H}{\partial y_k} \right\rangle = \delta_{jk} k_B T\right)$ to find the average energy per particle.

4. Explain briefly the phenomenon of Bose-Einstein condensation. Why is it that liquid He-4 forms a superfluid (at around $2.7K$), whereas liquid He-3 does not?

5. An ensemble of N spin-1/2 particles have the following composition as far as their spin states are concerned: $N/2$ of them in state $|S_z; + >$, $N/4$ of them in state $|S_z; - >$ and $N/4$ of them in state $|S_x; + >$. Write down the density matrix for this ensemble in the S_z basis. (Note that, $|S_x; + >= \frac{|S_z;+> + |S_z;->}{\sqrt{2}}$.)

6. A thermodynamic system has the fundamental equation, $S = \alpha(UVN)^{\frac{1}{3}}$. Find the equation of state relating pressure to volume, temperature and particle number.

7. The wave function for a four particle system is (incorrectly) given to be $\Psi(x_1, x_2, x_3, x_4) = \phi_a(x_1)\phi_b(x_2)\phi_c(x_3)\phi_d(x_4)$. If particles 1 and 2 are identical fermions and particle 3 and 4 are identical bosons, write the correct form of the wave function, Ψ. Justify your answer.

8. What is degeneracy temperature? Estimate the degeneracy temperature for a free electron gas having number density, $n = 10^{28}$ per cubic meter. (The thermal de Broglie wavelength is give by $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$.)

Part B

1. The frequency of an electromagnetic mode of wave vector $\vec{k}$ is $\omega_{\vec{k}} = ck$. In a box of volume V, there will be $\frac{2Vd^3k}{(2\pi)^3}$ such modes in a region d^3k surrounding a given wave vector. The Hamiltonian of the system is $\sum_{\vec{k}} h\nu_{\vec{k}} n_{\vec{k}}$, where $n_{\vec{k}}$ is the number of excitation in the mode with wave vector $\vec{k}$.

   (a) Derive the expression for the internal energy density, $\frac{U}{V}$. (You can leave it in terms of the integral over frequency.)

   (b) State and prove Stefan-Boltzmann law. (You need not evaluate the form of the Stefan-Boltzmann constant.)

[7 + 3]

2. Consider a dilute gas consisting of N linear molecules each with a permanent electric dipole moment, μ, contained in a vessel of volume V. The energy H_1 of a molecule in an electric field $\vec{E}$ pointing in the z direction is given by

$$H_1 = \frac{\vec{P}_{CM}^2}{2m} + \frac{p_\theta^2}{2I} + \frac{p_\phi^2}{2I\sin^2\theta} - \mu E\cos\theta \ ,$$

   where m is the mass of the molecule and I its moment of inertia.

   (a) Find the canonical partition function for this system. (Note that the state of a single molecule is given by specifying $\vec{R}_{CM}, \theta, \phi, \vec{P}_{CM}, p_\theta$ and p_ϕ.)

   (b) Determine the polarization $P \equiv \frac{N}{V}\langle \mu\cos\theta \rangle$.

[7 + 3]

3. Consider a system of two identical non interacting particles. The single particle energy levels of this system are two in number and have energies ϵ_1 and ϵ_2.

   (a) Find the partition function if the particle were classical distinguishable ones.

   (b) Find the partition function if particles were spin-1/2 fermions.

   (c) Find the partition function if the particles were spin-0 bosons.

   (d) For the case of bosons, find the average energy of the system.

[4 + 4 + 4 + 3]

Useful Formulas:

- Fermi-Dirac/Bose-Einstein distribution: $\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} \pm 1}$
- $dU = TdS - PdV + \mu dN$
- $h = 6.6 \times 10^{-34}\text{m}^2\ \text{Kgs}^{-1}$; $k_B = 1.4 \times 10^{-23}\ \text{JK}^{-1}$; $m_e = 9.1 \times 10^{-31}$ Kg