Birla Institute of Technology & Science – Pilani, K K Birla Goa Campus

First Semester 2022-2023

Statistical Mechanics Mid-semester Exam

PHY F312 October 31, 2022

Total Marks: 40 Duration: 90 Minutes

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1. A particular system obeys the equations $U = \frac{PV}{2}$ and $T = \left(\frac{DU^{3/2}}{VN^{1/2}}\right)^{1/2}$, where D is a positive constant.

   (a) Find the fundamental relation $S(U, V, N)$ for the system. (*Hint: You may have to use the fact that entropy is an extensive quantity to fix the fundamental relation completely, except possibly for a constant coming via integration.*)

   (b) Find chemical potential of the system as a function of T, V and N.

   (c) Find the Helmholtz free energy, $A(T, V, N)$.

[6 + 4 + 4]

2. Consider an ideal gas present in a cylindrical container of radius R and height, H ($H \ll$ radius of the earth) on the surface of the earth. The total number of particles in the gas in N and mass of each of the particles is m. The total energy of the gas is E.

   (a) Find the equilibrium distribution function, $f(\vec{p}, \vec{r})$, in the μ-space of the system. (You may leave the answer in terms of the Lagrange multiplier, β, appearing in the MB - distribution function.)

   (b) What is the average speed of the particles in the cylinder?

   (c) Is the average speed of the particles in in the upper half of the cylinder less than, equal to or more than the value found above? Justify.

   (d) Comment about the density distribution of the gas inside the cylinder for the cases when: (i) $\frac{E}{N} \ll mgH$ and (ii) $\frac{E}{N} \gg mgH$.

[6 + 4 + 2 + 2]

3. The relation between momentum and energy for an extreme relativistic particle is $\epsilon = pc$. Consider an ideal gas consisting of N extreme relativistic particles of mass m and total energy E confined to a volume, V.

   (a) Find the fundamental relation, $S(E, V, N)$ for the gas.

   (b) Find the pressure of the gas as as a function of energy density.

[3 + 3]

4. Let $\eta(\theta_1, \theta_2)$ be the efficiency of a Carnot engine working between two thermal reservoirs (with $\theta_1 > \theta_2$. Show that $\eta(\theta_1, \theta_2) = \eta(\theta_1, \theta_3)$ if and only if $\theta_2 = \theta_3$. You may not use the expression for η in terms of the absolute temperature of the reservoirs as that scale is set up making use of the above result. (*Hint: Look for violation of one of the statements of the second law of thermodynamics.*) **[6]**

Useful Formulae:

- $S = k_B \mathrm{Log}(\Omega)$
- $\underset{0 \le \sum_{i=1}^{M} r_i \le R}{\int \cdots \int} \Pi_{i=1}^{M} 4\pi r_i^2 dr_i = \frac{(8\pi R^3)^M}{(3M)!}$
- $\int_{-\infty}^{\infty} e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$
- $dU = TdS - PdV + \mu dN$
- Helmholtz free energy, $A = U - TS$
- Gibbs, free energy, $G = A + PV$
- Maxwell - Boltzmann distribution: $Ce^{-\beta(\frac{p^2}{2m} + V(\vec{r}))}$