BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI
FIRST SEMESTER 2023-24
PHY F312: *Statistical Mechanics*
Comprehensive Examination (Part B)
Open Book

Total marks: 55 Time: 120 mins

Useful Integrals

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} x^n e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5..(n-1)\pi^{1/2}}{2^{n/2} a^{(n+1)/2}} \qquad n = 0, 2, 4..$$

$$\int_{0}^{\infty} x^n e^{-ax^2} dx = \frac{\frac{n-1}{2}!}{2a^{(n+1)/2}} \qquad n = 1, 3, 5...$$

$$\mathcal{G}_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{z^{-1} e^x - 1}$$

$$\mathcal{F}_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{z^{-1} e^x + 1}$$

Ensembles

1. Consider a system of identical particles maintained at a temperature T and a chemical potential μ by a reservoir. Each particle can exist in two states with energies $\pm\epsilon$. Calculate the grand partition function, mean number of particles and mean number of particles occupying the two states. [5]

2. A particle can have 4 energies $\epsilon = -3\alpha/2, -\alpha/2, \alpha/2, 3\alpha/2$. Find

   (a) The single particle canonical partition function Z_1.

   (b) For N such distinguishable particles find the partition function Z_N.

   (c) Find mean energy and standard deviation in energy.

   (d) Find the specific heat C_V [10]

Fermions

3. Consider a 2 particle state $|a\ b\rangle$ where a and b are quantum numbers of the two particles. Show that the state $\frac{|a\ b\rangle + |b\ a\rangle}{\sqrt{2}}$ is inconsistent with Pauli exclusion principle but the state $\frac{|a\ b\rangle - |b\ a\rangle}{\sqrt{2}}$ is consistent with it. [2]

4. Consider a gas of N fermions at temperature T. Assume that the degeneracy of states

$$g(E)dE = \mathcal{D}dE$$

   where $\mathcal{D}$ is a constant.

   (a) At $T = 0$ write the expressions for N and hence obtain the expression for the Fermi energy E_F.

   (b) At $T = 0$ find the expression for total energy E in terms of E_F.

   (c) When $e^{\mu\beta} >> 1$ ignore the $+1$ in the denominator of the Fermi function and find and expression of N. Hence express the condition $e^{\mu\beta} >> 1$ (degenerate gas) as a condition on T and E_F.

   (d) Find the expression for Pressure of this gas at temperature T [2 + 2 + 4 + 5]

5. For an ideal Fermi gas of spin s particles at temperature T, show that when the temperature is high (small z) and $n = N/V$ is also small.

$$PV = NkT\left(1 + a\frac{N\lambda_{Th}^3}{V}\right)$$

What is the value of the constant a? You may use the approximations
$(\mathcal{F}_{5/2}(z) \approx z - \frac{z^2}{2^{5/2}} \quad \& \quad \mathcal{F}_{3/2}(z) \approx z - \frac{z^2}{2^{3/2}})$ [5]

Bosons

6. Consider an ideal Bose gas consisting of N particles at temperature T where the density of states $g(E)$ is given by $g(E)dE = \alpha V E^r dE$ where α is a constant. Calculate the (a) grand partition function (b) Calculate the temperature T_c at which BEC transition takes place (c) Is there BEC when $r = 0$? [6]

7. Using the second law of thermodynamics, Calculate the entropy, Helmholtz free energy, Gibbs free energy, and enthalpy of a photon gas in a cavity of volume V at temperature T. [6]

8. The dispersion relationship (relation between the frequency ω and the momentum p for waves in a solid is given by $\omega(p) = Ap^s$. What is the Debye cutoff temperature in D dimensions.

   For $D = 3$ how does specific heat of the solid depend on temperature at low temperatures and very high temperatures. [8]