Birla Institute of Technology & Science, Pilani
K. K. Birla Goa Campus
Theory of Relativity (PHY F315) SEM I 2022-23

30 December 2022 **Comprehensive Examination (Closed book)**
Time: 3 Hrs. **Max. Marks: 80** **Weight 40%**

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1. Using the Lorentz transformation equations *and* the fact that the spacetime interval $\Delta s^2 \equiv c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta^2 z$ is a Lorentz invariant, derive the formulas for length contraction and time dilation. **(10)**

2. At the LHC (Large Hadron Collider), two photons are detected with 4-momenta $P_1 = E_1/c(1,1,0,0)$ and $P_2 = E_2/c(1,\cos\theta,\sin\theta,0)$, respectively. Assuming that the photon pair resulted from the decay of a new particle ϕ such that $\phi \longrightarrow \gamma\gamma$, what is the mass of the new particle? **(10)**

3. The magnetic field inside a long solenoid carrying a steady current I and having n turns per unit length is given by $\vec{\boldsymbol{B}} = \mu_0\, n\, I\hat{\boldsymbol{x}}$, where $\hat{\boldsymbol{x}}$ is a unit vector parallel to the axis of the solenoid. Use this expression to find how the x-component of the magnetic field transforms under the standard Lorentz transformation. **(10)**

4. Show that $\sum_\mu D^{\mu\mu}$ and $\sum_\mu D_{\mu\mu}$ are not invariant under Lorentz transformations, but that $\sum_\mu \mathrm{D}_\mu{}^\mu$ is invariant. (Take $\mathbf{D}$ to be a tensor defined by its components $D^{\mu\nu}$.) Hint: For the case of $D^{\mu\mu}$ and $D_{\mu\mu}$, prove by taking specific examples. **(10)**

5. Prove that the Kronecker delta, $\delta^\mu{}_\nu$, is a tensor, under a general coordinate transformation. **(10)**

6. Derive the relation between the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ and the metric tensor $g_{\mu\nu}$ from the following conditions: (i) $\nabla_\lambda g_{\mu\nu} = 0$, where ∇_λ is the covariant derivative, and (ii) $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$. (The result is the so-called "fundamental theorem" in Riemannian geometry.)
   Hint: 1. If
   $$\nabla_\lambda T_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial x^\lambda} - \sum_\rho \Gamma^\rho_{\lambda\mu} T_{\rho\nu} - \sum_\rho \Gamma^\rho_{\lambda\nu} T_{\mu\rho},$$
   what are the other two independent equations?
   2. $\sum_\nu g^{\mu\nu} g_{\nu\alpha} = \delta^\mu{}_\alpha$. **(10)**

7. On the surface of a unit two-dimensional sphere, $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.
   (a) Find all the non-zero Christoffel symbols (connection coefficients).
   (b) A vector $\mathbf{A}$ is equal to $\hat{\boldsymbol{\theta}}$ at $\theta = \theta_0, \phi = 0$. It is then parallel transported around the circle $\theta = \theta_0$. What is the equation of parallel transport for $\mathbf{A}$ along the ϕ coordinate line (θ=constant)?
   (c) Solve the equation of parallel transport, from the previous section, for the components A_θ and A_ϕ.
   (d) Use the previous result to show that $\mathbf{A}$ after it is parallel transported around the circle $\theta = \theta_0$ (i. e., at $\phi = 2\pi$), is not equal to $\mathbf{A}$ at $\phi = 0$. What is the magnitude of $\mathbf{A}$ at $\phi = 2\pi$? **(5+5+5+5=20)**

Note:

$$T'^{\mu_1\mu_2\ldots\mu_m}_{\alpha_1\alpha_2\ldots\alpha_n} = \sum_{\nu_1,\nu_2,\ldots,\nu_m,\beta_1,\beta_2,\ldots,\beta_n} \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}}\frac{\partial x'^{\mu_2}}{\partial x^{\nu_2}}\cdots\frac{\partial x'^{\mu_m}}{\partial x^{\nu_m}}\frac{\partial x^{\beta_1}}{\partial x'^{\alpha_1}}\frac{\partial x^{\beta_2}}{\partial x'^{\alpha_2}}\cdots\frac{\partial x^{\beta_n}}{\partial x'^{\alpha_n}} T^{\nu_1\nu_2\ldots\nu_m}_{\beta_1\beta_2\ldots\beta_n}.$$

$$\Gamma^i_{jk} = \frac{1}{2}\sum_l g^{il}\left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right);\ \text{Parallel Transport: } \frac{\mathrm{d}v^i}{\mathrm{d}u} = -\sum_{j,k}\Gamma^i_{jk} v^j \frac{\mathrm{d}x^k}{\mathrm{d}u}.$$

****************************** That's all Folks! ******************************